(b) Must every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ have a fixed point? [Hint: Consider the upper half-plane.]

13. The **pseudo-hyperbolic distance** between two points $z, w \in \mathbb{D}$ is defined by

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|.$$

(a) Prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$\rho(f(z), f(w)) \le \rho(z, w) \qquad \text{for all } z, w \in \mathbb{D}.$$

Moreover, prove that if f is an automorphism of $\mathbb{D}$ then f preserves the pseudo-hyperbolic distance

$$\rho(f(z), f(w)) = \rho(z, w) \qquad \text{for all } z, w \in \mathbb{D}.$$

[Hint: Consider the automorphism $\psi_\alpha(z) = (z - \alpha)/(1 - \overline{\alpha}z)$ and apply the Schwarz lemma to $\psi_{f(w)} \circ f \circ \psi_w^{-1}$.]

(b) Prove that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2} \qquad \text{for all } z \in \mathbb{D}.$$

This result is called the Schwarz-Pick lemma. See Problem 3 for an important application of this lemma.

14. Prove that all conformal mappings from the upper half-plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ take the form

$$e^{i\theta} \frac{z - \beta}{z - \overline{\beta}}, \qquad \theta \in \mathbb{R} \text{ and } \beta \in \mathbb{H}.$$

15. Here are two properties enjoyed by automorphisms of the upper half-plane.

(a) Suppose Φ is an automorphism of $\mathbb{H}$ that fixes three distinct points on the real axis. Then Φ is the identity.

(b) Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two pairs of three distinct points on the real axis with

$$x_1 < x_2 < x_3 \qquad \text{and} \qquad y_1 < y_2 < y_3.$$

Prove that there exists (a unique) automorphism Φ of $\mathbb{H}$ so that $\Phi(x_j) = y_j$, $j = 1, 2, 3$. The same conclusion holds if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$.